swissQuant
Mastering Complexity

Mathematical Challenge March 2017

Entropy Pooling

References

- [1] Meucci, A. "Fully Flexible Views: Theory and Practice"
- [2] Meucci, A. "The Black-Litterman Approach: Original Model and Extensions"

Description

Introduction

In this mathematical challenge we present the Entropy Pooling approach, introduced by A. Meucci in [1] as a generalization of the Black-Litterman approach. First we will briefly summarize the latter, and then introduce the former as its natural extension.

Black-Litterman Approach

The model assumes that the market contains N securities which have normally distributed returns:

$$X \sim N(\mu, \Sigma)$$

with the covariance estimated from historical observations. Forecasting the average can be non-trivial and for this reason μ is modelled itself as a random variable:

$$\mu \sim N(\pi, \tau\Sigma)$$

where π is a "best guess" on μ.

Views

Views in Black-Litterman are statements on the expectation of the market, which in the normal case consists in a statement on the vector μ. We define a pick matrix P, which defines the weight of each expected return in the specific view. To include uncertainty in the views, Black-Litterman uses a normal model:

$$P\mu \sim N(v, \Omega)$$

It is then possible to prove that the posterior distribution for the market factors incorporating the views is given by:

$$\mu_{BL} = \pi + \tau\Sigma P'(\tau P\Sigma P' + \Omega)^{-1}(v - P\pi)$$

$$\Sigma_{BL} = (1+\tau)\Sigma - \tau^2\Sigma P'\left(\tau P\Sigma P' + \Omega\right)^{-1} P\Sigma$$

This parameters can then be used in a mean-variance optimizer to derive the optimal allocation for a portfolio.

Entropy Pooling

Meucci extended the Black-Litterman approach to overcome multiple limitations of the model, e.g. the normality assumption for the returns and the linearity of the views.

Reference Model

Meucci assumes the existence of a reference model to describe the joint distribution of risk factor via their density function:

$$X \sim f_X$$

Views

In the entropy pooling approach, the views are generic functions of the risk factors $g_1(X), g_2(X), \dots g_K(X)$, which are not necessarily linear. The functions generate a K-dimensional random variable with a joint distribution implied by the reference model:

$$V = g(X) \sim f_V$$

The posterior distribution

The views create a distortion of the reference model. The posterior distribution should be such that it satisfies the views, but it is not too “far away” from the reference model itself. Defining the relative entropy of a distribution $\tilde{f}_X$ with respect to a distribution f_X as:

$$\mathcal{E}(\tilde{f}_X, f_X) = \int \tilde{f}_X(x)[\ln \tilde{f}_X(x) - \ln f_X(x)]\, dx$$

we can measure how much $\tilde{f}_X$ is distorted w.r.t f_X and how much information contains. We define the posterior distribution as:

$$\tilde{f}_X = \underset{f \in V}{\operatorname{argmin}}\ \mathcal{E}(f, f_X)$$

As a special case, if the market is normally distributed $X \sim N(\mu, \Sigma)$, and the views are given on combination of expectations and covariances as:

$$\begin{cases} E(QX) = \tilde{\mu}_Q \\ Cov(GX) = \tilde{\Sigma}_G \end{cases}$$

then it is possible to prove that also the posterior distribution is normally distributed $X \sim N(\tilde{\mu}, \tilde{\Sigma})$, where

$$\tilde{\mu} = \mu + \Sigma Q'(Q\Sigma Q')^{-1}(\tilde{\mu}_Q - Q\mu)$$
$$\tilde{\Sigma} = \Sigma + \Sigma G'\left((G\Sigma G')^{-1}\tilde{\Sigma}_G(G\Sigma G')^{-1} - (G\Sigma G')^{-1}\right)G\Sigma$$

Questions:

Q1: Assume that $c(x)$ is the copula of a distribution function f, f_n is the n –th marginal and F_n is the cumulative distribution function of the n –th marginal. Prove that the relative entropy between f and the prior $\underline{f}$ can be written as:

$$\mathcal{E}\left(f,\underline{f}\right) = \sum_{n=1}^{N} \mathcal{E}\left(f_n,\underline{f_n}\right) + \mathcal{E}(c,\underline{c}) + \mathcal{E}_{CR}$$

where

$$\mathcal{E}_{CR} = E\left[\ln \frac{c(U_1,\dots,U_N)}{c\left(\underline{F_1}\left(F_1^{-1}(U_1)\right),\dots,\underline{F_N}\left(F_N^{-1}(U_1)\right)\right)}\right]$$

What is the challenge behind using this kind of structure?

Q2: Start from the assumption that a n –dimensional market follows a linear factor model of the form

$$X = bZ + U$$

where Z is a k-dimensional set of risk factors, with $k \ll n$, which follows a normal distribution. Write down the entropy-pooling optimization problem for this case and prove that the entropy is not convex. Therefore, what would simplify a numerical solution of the problem in this case?

Q3: Entropy pooling can be formulated in a parametric way, assuming a distribution, and a non-parametric way. Derive the formula for the non-parametric case and explain the differences between the two approaches in terms of advantages and disadvantage. For which applications is each method more appropriate?

We look forward to your opinions and insights.

Best Regards,

swissQuant Group Leadership Team