swissQuant
Mastering Complexity

Mathematical Challenge February 2020

Game options

References

[1] E. B. Dynkin, “Game variant of a problem on optimal stopping", Soviet Mathematics-Doklady, vol. 10, pp. 270-274, 1969.

[2] Y. Kifer, “Game options", Finance and Stochastics, vol. 4, no. 4, pp. 443-463, 2000.

Description

Motivation

A European Option is a contingent claim that may be exercised by the buyer only at expiry of the contract, i.e. at a single predefined point in time. An American Option on the other hand may be exercised by the buyer at any time before expiry.

Israeli Options (or Game Options) [2] are extension of American Options. They enable both their buyer and issuer to exercise them at any time before expiry. In general, if the contract is terminated by the issuer a certain penalty must be paid to the buyer. The pricing of those derivatives is based on an extension of the optimal stopping theory called Dynkin's games [1].

Technical Details

The general setup consist of a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions and two $\mathbb{F}$-adapted right continuous with left limits stochastic payoffs X_t and Y_t such that $X_t \geq Y_t \geq 0$.

If the buyer exercises the option at time τ he will receive the amount Y_τ. In case the issuer exercises at time σ he will have to pay the buyer the amount X_σ. The difference $\delta_t =$

$X_t - Y_t \geq 0$ is interpreted as the penalty the issuer pays to the buyer for the contract cancellation. The option's payoff can be written as

$$Z_{\tau,\sigma} = Y_\tau \mathbb{1}_{\tau \leq \sigma} + X_\sigma \mathbb{1}_{\sigma < \tau}. \tag{1}$$

Note that it is assumed that the buyer has priority in case both exercise at the same time.

The simplest examples are the Israeli-Call Option with payoffs

$$Y_t = (S_t - K)^+ \quad \text{and} \quad X_t = (S_t - K)^+ + \delta \tag{2}$$

and the Israeli-Put Option with payoffs

$$Y_t = (K - S_t)^+ \quad \text{and} \quad X_t = (K - S_t)^+ + \delta \tag{3}$$

for some risky underlying S_t and constant penalty δ.

Assuming a Black-Scholes framework, that is constant interest rate r and the risky asset to follow a geometric Brownian motion under the risk neutral measure the following theorem hold.

Theorem (Kifer) Suppose that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} e^{-rT} X_t \mid \mathcal{F}_0\right] < \infty \tag{4}$$

and if $T = \infty$ that

$$\mathbb{P}\left(\lim_{t \to \infty} e^{-rt} X_t = 0\right) = 1. \tag{5}$$

Let $\mathcal{T}_{t,T}$ be the class of $\mathbb{F}$-stopping time values in $[t, T]$. The value of the Israeli option under the Black-Scholes framework is given by $V = \{V_t : t \in [0, T]\}$ where

$$V_t = \sup_{\tau \in \mathcal{T}_{t,T}} \inf_{\sigma \in \mathcal{T}_{t,T}} \mathbb{E}[e^{-r(\tau \wedge \sigma - t)} Z_{\tau,\sigma} | \mathcal{F}_t] \tag{6}$$

$$= \inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}[e^{-r(\tau \wedge \sigma - t)} Z_{\tau,\sigma} | \mathcal{F}_t]. \tag{7}$$

Where $\tau \wedge \sigma = \min(\tau, \sigma)$. Further, the optimal stopping strategies for the buyer and issuer, respectively are

$$\tau^* = \inf\{t \in [0, T] : Y_t \geq V_t\} \wedge T \quad \text{and} \quad \sigma^* = \inf\{t \in [0, T] : X_t \leq V_t\} \wedge T. \tag{8}$$

Questions

- **Q1:** *Why can an Israeli Option can be sold cheaper than an American Option?*
- **Q2:** *Give a numerical scheme for the price computation in discrete time.*

- **Q3:** *What are the prices of Israeli-Call and -Put Options if the constant penalty is zero,* $\delta = 0$*?*
- **Q4:** *Assume a constant penalty* $\delta > \sup_{0 \le \tau \le T} \mathbb{E}[Y_\tau | \mathcal{F}_\tau]$*. Show that this case reduce to an American option.*
- **Q5:** *Is the above mentioned theorem still valid for deterministic interest rates and generally* $\mathbb{F}$*-adapted payoffs* X_t *and* Y_t *with* $X_t \ge Y_t \ge 0$*?*

We look forward to your opinions and insights.

Best Quant Regards,

swissQuant Group Leadership Team